

Approximations solution for Volterra integro-differential equations

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Abstract— The goal of this work is to we present new approach for the numerical solution of Volterra integro-differential equations of the second order by the means of a new technical method inspired from Taylor series method and regularization method using a modified Simpson technique. We treat some examples for the reason to illustrate accuracy and efficiency of this method.

Keywords— Volterra integro-differential equations, Derivation of Volterra integrals, Integral method, Modified Simpson technique.

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Solución de aproximaciones para ecuaciones integro-diferenciales de Volterra

Resumen— El objetivo de este trabajo es presentar un nuevo enfoque para la solución numérica de ecuaciones integro-diferenciales de Volterra de segundo orden mediante un nuevo método técnico inspirado en el método de las series de Taylor y el método de regularización utilizando una técnica de Simpson modificada. Se presentan algunos ejemplos para ilustrar la precisión y la eficiencia de este método.

Palabras clave— Ecuaciones integro-diferenciales de Volterra, derivación de integrales de Volterra, método integral, Técnica de Simpson modificada.

1 Introduction

Volterra Integro-differential equations can be used to model many mathematical problems, used by engineers, physicists and applied mathematicians. Noting that the analytical solution of these equations are almost impossible, for this reason numerous numerical methods have been developed to solve Volterra Integro-differential equations. The authors in [1] used Adomian decomposition method to solve numericaly Fredholm integro-differential equations. In [2]-[3] authors studied Euler and Chebyshev polynomials method for solving linear integro-differential equations. Finite element Method and variational Iteration Technique for solving Volterra and fuzzy Volterra integro-differential equations in [4], [5] respectively. [6] – [7] used a variational form with Bernoulli series and a variational form with Legendre series for linear integral equations. The author in [8] use the spline function for the numerical solution of weakly singular integro-differential equations. The Block Pulse Functions and Operational Matrices are used in [9]. A Comparison between the variational Iteration Method and trapezoidal rule for solving linear integro-differential equations are studied in [10]. The authors in [11] extend the optimal auxiliary function method to integro-differential equations. This method has the auxiliary function and

convergence control parameters that accelerate the method's convergence.

Given the initial value of the second order

$$\begin{aligned} u''(x) + a_1(x)u'(x) + a_0(x)u(x) \\ - \int_{\Omega} k(x,t)u(t)dt = f(x), \end{aligned} \quad (1)$$

With its initial conditions (2)

$$u(x_0) = \alpha, \quad u'(x_0) = \beta. \quad (2)$$

Where, the functions $f(x)$, $a_0(x)$, $a_1(x)$ and $k(x,t)$ are given and continuous functions in Ω and $\Omega \times \Omega$, respectively, the function $u(t)$ is to be determined as continuous function depending on $\Omega = [a, x]$ for a Volterra integro-differential equation or $\Omega = [a, b]$ for a Fredholm integro-differential equation.

The goal of this is problem is to finding the solution $u(x)$ of the equation (1) that satisfies an initial condition (2), where x_0 is some fixed value of point and $u(x_0)$ is a fixed state. We write this initial value problem with integral equation (IVPIE) concisely as:

$$\begin{cases} u''(x) + a_1(x)u'(x) + a_0(x)u(x) \\ - \int_{\Omega} k(x,t)u(t)dt = f(x), \\ u(x_0) = \alpha, \quad u'(x_0) = \beta. \end{cases} \quad (3)$$

The initial conditions usually picks out a specific value of the arbitrary constants C_1 and C_2 that appears in the general solution of the equation. So, it selects one of the many possible states that satisfy the differential equation.

2 Numerical Methods of IVPIE

Suppose we solve the following initial value problem with integral equation (3) on the interval Ω

$$\begin{cases} u''(x) + a_1(x)u'(x) + a_0(x)u(x) - \int_{\Omega} k(x,t)u(t)dt = f(x), \\ u(x_0) = \alpha, \quad u'(x_0) = \beta. \end{cases}$$

If the functions $a_0(x)$ and $a_1(x)$ are continuous on the interval $\Omega = [a, b]$ or $\Omega = [a, x]$ containing the point $x = x_0$. Then there exists a unique solution $u(x)$ of the problem (3) if its integral is of Volterra, whereas if its integral is of Fredholm, we must add an appropriate conditions for the existence of the solution. Conversely, with the discontinuity of $a_0(x)$ or $a_1(x)$ the existence nor the uniqueness of a solution is guaranteed.

Seeking a continuous solution defined at each point x , we develop a strategy of discretization the problem to determine an approximation at discrete points in the interest interval. Therefore, the idea is to replace the continuous function model (3) with an approximate discrete function model that is amenable to computer solution.

We divide the interval $a \leq x \leq b$ into N segments of constant length h , called the step size. Thus the step size is $h = \frac{b-a}{N}$. This defines a set of equally spaced discrete points $a = x_0 \leq x_1 \leq x_2 \leq x_3 \dots, x_N = b$, where $x_{i+1} = x_i + h$, $i = 0, 1, 2, \dots, N$. Now, suppose we know the solution $u(x_i)$ of the initial value problem at the point x_i . How could we estimate the solution at the point x_{i+1} ?

$$\begin{aligned} u(x_0) &= \alpha, \quad u'(x_0) = \beta \\ u'(x) &\approx \frac{u(x) - u(x-h)}{h} \end{aligned} \quad (4)$$

$$\begin{aligned} u''(x) &= f(x) - a_1(x)u'(x) - a_0(x)u(x) \\ &+ \int_{\Omega} k(x,t)u(t)dt \end{aligned} \quad (5)$$

$$\begin{aligned} u'''(x) &= (f(x) - a_1(x)u'(x) - a_0(x)u(x))' \\ &+ \left(\int_{\Omega} k(x,t)u(t)dt \right)' \end{aligned} \quad (6)$$

where the derivative $\left(\int_{\Omega} k(x,t)u(t)dt \right)'$ if $\Omega = [a, x]$ is given by

$$\begin{aligned} \left(\int_a^x k(x,t)u(t)dt \right)' &= k(x,x)u(x) \\ &+ \int_a^x \frac{\partial}{\partial x} k(x,t)u(t)dt \end{aligned} \quad (7)$$

And the derivative $\left(\int_{\Omega} k(x,t)u(t)dt \right)'$ if $\Omega = [a, b]$ is given as (8).

$$\left(\int_a^b k(x,t)u(t)dt \right)' = \int_a^b \frac{\partial}{\partial x} k(x,t)u(t)dt, \quad (8)$$

3 Modified Taylor series method of IVPVI

$$\begin{aligned} u(x_{i+1}) &= u(x_i) + hu'(x_i) + \frac{h^2}{2}u''(x_i) \\ &+ \frac{h^3}{3!}u'''(x_i) + \frac{h^4}{4!}u^{(4)}(\omega_i); \quad x_i < \omega_i < x_{i+1}. \end{aligned} \quad (9)$$

For the expression (7) the truncation error, due to the terms omitted from the series, is

$$E = \frac{h^4}{4!}u^{(4)}(\omega_i); \quad x_i < \omega_i < x_{i+1}. \quad (10)$$

It's easy to see that we can obtain the value of $u(x_{i+1})$ from the value of $u(x_i)$ using the relations (4), (5), (6), and (7) or (8).

4 Regularization method using modified Simpson method

$$\begin{aligned} \int_a^x u''(x)dx &= \int_a^x f(x)dx - \int_a^x a_1(x)u'(x)dx \\ &- \int_a^x a_0(x)u(x)dx + \int_a^x \int_{\Omega} k(t,x)u(t)dt dx, \end{aligned} \quad (11)$$

We get

$$\begin{aligned} u'(x) &= \int_a^x f(t)dt - a_1(x)u(x) \\ &+ a_1(a)u(a) + \int_a^x (a_1(x) - a_0(x))u(x)dx \\ &+ \int_a^x \int_{\Omega} k(t,x)u(t)dt dx + \beta, \end{aligned} \quad (12)$$

In addition, after integration, we obtain

$$\begin{aligned} u(x) &= \int_a^x (x-t)f(t)dt + \int_a^x B(t)u(t)dt \\ &+ \int_a^x k_1(t,x)u(t)dt + \gamma(x-a) + \alpha, \end{aligned} \quad (13)$$

Where:

$$\begin{aligned} B(t) &= (x-t)(a_1(t) - a_0(t)) - a_1(t), \\ \gamma &= a_1(a)u(a) + \beta \quad \text{and} \quad k_1(t,x) = \int_t^x \left(\int_{\Omega} k(t,z)dz \right) dx. \end{aligned}$$

For the integral equation (9) using spectral methods, the error represents the difference between the value $u(x)$ of the integral and its numerical result is

$$E = \frac{\varphi(x)}{(N+1)!} \sup |f^{(N+1)}(\omega)|$$

Where $\varphi(x) = \prod_{i=1}^N (x - x_i)$ is the monic polynomial of degree N with roots at $x_1, x_2, \dots, x_N$ and h is the "step length" given by $h = \frac{b-a}{N}$.

Denoting by u_i the approximation of the solution $u(x_i)$ at $x = x_i$. By algorithm and the given value $u_0 = u(x_0) = \alpha$ of the solution at the point x_0 we can calculate the approximations solutions u_i , $i = 1, 2, \dots, N$.

5 Numerical examples

5.1 Example 1

Solve the initial value problem with Volterra integral:

$$\begin{cases} u''(x) + 3u(x) - \int_0^x \sin(x+t)u(t)dt = f(x), \\ u(0) = 0, u'(0) = -2. \end{cases}$$

$$f(x) = (x^2 - 2x - 2)\cos 2x + (2 - 2x)\sin 2x + 2\cos x - 2\sin x + 3x^2 - 6x + 2,$$

From $x = 0$ to 1 where the analytical solution of the problem is:

$$u(x) = x^2 - 2x$$

Modified Taylor series method of IVPVI:

$$\begin{cases} u''(x) = f(x) - 3u(x) + \int_0^x \sin(x+t)u(t)dt, \\ u(0) = 0, u'(0) = -2. \end{cases}$$

$$u''(0) = 0,$$

$$\begin{cases} u'''(x) = (f(x))' - 3u'(x) \\ + \left(\int_0^x \sin(x+t)u(t)dt \right)', \\ u(0) = 0, u'(0) = -2, u''(0) = 0. \end{cases}$$

Which is approximated by:

$$u(x_{i+1}) \approx u(x_i) + hu'(x_i) + \frac{h^2}{2}u''(x_i) + \frac{h^3}{3!}u'''(x_i)$$

Table 1
Modified Taylor series method of order three for example 1

Val of x	Exact sol	Approx sol	Error
0,00e+00	0,00e+00	0,00e+00	0,00e+00
2,50e-01	-4,37e-01	-4,37e-01	4,63e-05
5,00e-01	-7,50e-01	-7,50e-01	8,43e-05
7,50e-01	-9,37e-01	-9,37e-01	1,08e-05
1,00e+00	-1,00e+00	-1,00e+00	1,14e-05

Source: The authors.

Regularization Method using modified Simpson method:

$$\begin{cases} u''(x) + 3u(x) - \int_0^x \sin(x+t)u(t)dt = f(x), \\ u(0) = 0, u'(0) = -2. \end{cases}$$

$$f(x) = (x^2 - 2x - 2)\cos 2x + (2 - 2x)\sin 2x + 2\cos x - 2\sin x + 3x^2 - 6x + 2,$$

After integrating twice, we obtain a Volterra integral equation; using the modified Simpson method, we get the approximate solution.

Table 2
Regularization method using modified Simpson method for example 1

Val of x	Exact sol	Approx sol	Error
0,00e+00	0,00e+00	1,52e-15	1,52e-15
2,50e-01	-4,37e-01	-4,37e-01	1,03e-07
5,00e-01	-7,50e-01	-7,50e-01	1,87e-07
7,50e-01	-9,37e-01	-9,37e-01	2,40e-07
1,00e+00	-1,00e+00	-1,00e+00	2,56e-07

Source: The authors.

5.2 Example 2

Solve the initial value problem with Volterra integral

$$\begin{cases} u''(x) + u(x) + \int_0^x x \tan x u(t)dt = f(x), \\ u(0) = 1, u'(0) = 0. \end{cases}$$

$$f(x) = x(1 - \cos x),$$

From $x = 0$ to 1 where the analytical solution of the problem is

$$u(x) = \cos x$$

Modified Taylor series method of IVPVI

$$\begin{cases} u''(x) = f(x) - u(x) - \int_0^x x \tan x u(t)dt, \\ u(0) = 1, u'(0) = 0. \end{cases}$$

$$u''(0) = -1,$$

$$\begin{cases} u'''(x) = (f(x))' - u'(x) \\ - \left(\int_0^x x \tan x u(t)dt \right)', \\ u(0) = 1, u'(0) = 0, u''(0) = -1. \end{cases}$$

Which is approximated by:

$$u(x_{i+1}) \approx u(x_i) + hu'(x_i) + \frac{h^2}{2}u''(x_i) + \frac{h^3}{3!}u'''(x_i)$$

Table 3

Modified Taylor series method of order three for example 2

Val of x	Exact sol	Approx sol	Error
0,00e+00	1,00e+00	1,00e+00	0,00+00
2,50e-01	9,68e-01	8,03e-01	1,07e-04
5,00e-01	8,77e-01	6,84e-01	2,09e-04
7,50e-01	7,31e-01	6,12e-01	2,96e-04
1,00e+00	5,40e-01	5,67e-01	3,62e-04

Source: The authors.

Regularization method using modified Simpson method:

$$\begin{cases} u''(x) + u(x) + \int_0^x x \tan x u(t) dt = f(x), \\ u(0) = 1, u'(0) = 0. \end{cases}$$

$$f(x) = x(1 - \cos x),$$

After integrating twice, we obtain a Volterra integral equation; using the modified Simpson method, we get the approximate solution.

Table 4

Regularization method using modified Simpson method for example 2

Val of x	Exact sol	Approx sol	Error
0,00e+00	1,00e+00	1,00e+00	7,50e-13
2,50e-01	9,68e-01	9,68e-01	1,58e-07
5,00e-01	8,77e-01	8,77e-01	2,78e-07
7,50e-01	7,31e-01	7,31e-01	3,41e-07
1,00e+00	5,40e-01	5,40e-01	3,35e-07

Source: The authors.

5.3 Example 3

Solve the initial value problem with Volterra integral:

$$\begin{cases} u''(x) + u'(x) + u(x) - \int_0^x e^{-(x+t)} u(t) dt = f(x), \\ u(0) = 1, u'(0) = 0. \end{cases}$$

$$f(x) = \frac{1}{2}e^{-x}(e^{-2x} - 1) - e^{-x},$$

From $x = 0$ to 1 where the analytical solution of the problem is

$$u(x) = e^{-x}$$

Modified Taylor series method of IVPVI:

$$\begin{cases} u''(x) = f(x) - u'(x) - u(x) + \int_0^x e^{-(x+t)} u(t) dt, \\ u(0) = 1, u'(0) = -1. \end{cases}$$

$$u''(0) = 1,$$

$$\begin{cases} u'''(x) = (f(x))' - u'(x) - u(x) + \int_0^x e^{-(x+t)} u(t) dt \\ - \left(\int_0^x e^{-(x+t)} u(t) dt \right)', \\ u(0) = 1, u'(0) = -1, u''(0) = 1. \end{cases}$$

Which is approximated by:

$$u(x_{i+1}) \approx u(x_i) + hu'(x_i) + \frac{h^2}{2}u''(x_i) + \frac{h^3}{3!}u'''(x_i)$$

Table 5

Modified Taylor series method of order three for example 3

Val of x	Exact sol	Approx sol	Error
0,00e+00	1,00e+00	1,00e+00	8,88e-16
2,50e-01	7,78e-01	7,78e-01	6,67e-04
5,00e-01	6,06e-01	6,05e-01	1,15e-03
7,50e-01	4,72e-01	4,70e-01	1,47e-03
1,00e+00	3,67e-01	3,66e-01	1,65e-03

Source: The authors.

Regularization method using modified Simpson method:

$$\begin{cases} -u''(x) + u'(x) + u(x) - \int_0^x e^{-(x+t)} u(t) dt = f(x), \\ u(0) = 1, u'(0) = 0. \end{cases}$$

$$f(x) = \frac{1}{2}e^{-x}(e^{-2x} - 1) - e^{-x},$$

After integrating twice, we obtain a Volterra integral equation; using the modified Simpson method, we get the approximate solution.

Table 6

Regularization method using modified Simpson method for example 3

Val of x	Exact sol	Approx sol	Error
0,00e+00	1,00e+00	1,00e+00	1,39e-14
2,50e-01	7,78e-01	7,78e-01	5,89e-07
5,00e-01	6,06e-01	6,06e-01	9,01e-07
7,50e-01	4,72e-01	4,72e-01	1,05e-06
1,00e+00	3,67e-01	3,67e-01	1,12e-06

Source: The authors.

5.4 Example 4

Solve the initial value problem with Volterra integral:

$$\begin{cases} u''(x) + u'(x) + xu(x) + \int_0^x x^2 t u(t) dt = f(x), \\ u(0) = 0, u'(0) = 1. \end{cases}$$

$$f(x) = (x^4 - 2x^3 + 3x^2 + 2x + 3)e^x - 2x^2,$$

From $x = 0$ to 1 where the analytical solution of the problem is

$$u(x) = xe^x$$

Modified Taylor v method of IVPVI:

$$\begin{cases} u''(x) = f(x) - u'(x) - xu(x) - \int_0^x x^2 t u(t) dt, \\ u(0) = 0, u'(0) = -1. \end{cases}$$

$$u''(0) = 2,$$

$$\begin{cases} u'''(x) = (f(x))' - u'(x) - (xu(x))' \\ - \left(\int_0^x x^2 t u(t) dt \right)', \\ u(0) = 0, u'(0) = 1, u''(0) = 2. \end{cases}$$

Which is approximated by:

$$u(x_{i+1}) \approx u(x_i) + hu'(x_i) + \frac{h^2}{2} u''(x_i) + \frac{h^3}{3!} u'''(x_i)$$

Table 7

Modified Taylor series method of order three for example 4

Val of x	Exact sol	Approx sol	Error
0,00e+00	0,00e+00	0,00e+00	0,00e+00
2,50e-01	3,21e-01	2,01e-01	3,84e-05
5,00e-01	8,24e-01	3,94e-01	2,37e-05
7,50e-01	1,58e+00	5,71e-01	4,18e-05
1,00e+00	2,71e+00	7,24e-01	1,89e-05

Source: The authors.

Regularization Method using modified Simpson method:

$$\begin{cases} u''(x) + u'(x) + xu(x) + \int_0^x x^2 t u(t) dt = f(x), \\ u(0) = 0, u'(0) = 1. \end{cases}$$

$$f(x) = (x^4 - 2x^3 + 3x^2 + 2x + 3)e^x - 2x^2,$$

After integrating twice, we obtain a Volterra integral equation; using the modified Simpson method, we get the approximate solution.

Table 8

Regularization method using modified Simpson method for example 4

Val of x	Exact sol	Approx sol	Error
0,00e+00	0,00e+00	8,75e-14	8,75e-14
2,50e-01	3,21e-01	2,01e-01	4,47e-07
5,00e-01	8,24e-01	3,94e-01	8,70e-07
7,50e-01	1,58e+00	5,71e-01	1,03e-06
1,00e+00	2,71e+00	7,24e-01	4,45e-07

Source: The authors.

5.5 Example 5

Solve the initial value problem with Volterra integral:

$$\begin{cases} u''(x) + u'(x) + u(x) - \int_0^x xt u(t) dt = f(x), \\ u(0) = 0, u'(0) = 1. \end{cases}$$

$$\begin{aligned} f(x) &= \left(1 + \frac{1}{2}x - \frac{1}{2}x^3 \right) \ln(x+1) \\ &+ \frac{x}{(x+1)^2} + \frac{1}{4}x^3 - \frac{1}{2}x^2, \end{aligned}$$

From $x = 0$ to 1 where the analytical solution of the problem is

$$u(x) = \ln(x+1)$$

Modified Taylor series method of IVPVI

$$\begin{cases} u''(x) = f(x) - u'(x) - u(x) + \int_0^x xt u(t) dt, \\ u(0) = 0, u'(0) = 1. \end{cases}$$

$$u''(0) = -1,$$

$$\begin{cases} u'''(x) = (f(x))' - u''(x) - u'(x) \\ + \left(\int_0^x xt u(t) dt \right)', \\ u(0) = 0, u'(0) = 1, u''(0) = -1. \end{cases}$$

Which is approximated by:

$$u(x_{i+1}) \approx u(x_i) + hu'(x_i) + \frac{h^2}{2} u''(x_i) + \frac{h^3}{3!} u'''(x_i)$$

Table 9

Modified Taylor series method of order three for example 5

Val of x	Exact sol	Approx sol	Error
0,00e+00	0,00e+00	0,00e+00	0,00e+00
2,50e-01	2,23e-01	2,23e-01	2,16e-05
5,00e-01	4,05e-01	4,05e-01	1,58e-05
7,50e-01	5,59e-01	5,59e-01	2,25e-05
1,00e+00	6,93e-01	6,93e-01	7,95e-06

Source: The authors.

Regularization Method using modified Simpson method:

$$\begin{cases} u''(x) + u'(x) + u(x) - \int_0^x xt u(t) dt = f(x), \\ u(0) = 0, u'(0) = 1. \end{cases}$$

$$\begin{aligned} f(x) &= \left(1 + \frac{1}{2}x - \frac{1}{2}x^3 \right) \ln(x+1) \\ &+ \frac{x}{(x+1)^2} + \frac{1}{4}x^3 - \frac{1}{2}x^2, \end{aligned}$$

After integrating twice, we obtain a Volterra integral equation; using the modified Simpson method, we get the approximate solution.

Table 10
Regularization method using modified Simpson method for example 5

Val of x	Exact sol	Approx sol	Error
0,00e+00	0,00e+00	2,42e-15	2,42e-15
2,50e-01	2,23e-01	2,23e-01	2,29e-07
5,00e-01	4,05e-01	4,05e-01	3,64e-07
7,50e-01	5,59e-01	5,59e-01	4,60e-07
1,00e+00	6,93e-01	6,93e-01	5,28e-07

Source: The authors.

5.6 Example 6

Solve the initial value problem with Volterra integral:

$$\begin{cases} u''(x) + xu'(x) + xu(x) - \int_0^x (x^2 - t)u(t)dt = f(x), \\ u(0) = 0, u'(0) = 1. \end{cases}$$

$$f(x) = x^2 \cos x + x \sin x - x^2,$$

from $x = 0$ to 1 where the analytical solution of the problem is

$$u(x) = \sin x$$

Modified Taylor series method of IVPVI:

$$\begin{cases} u''(x) = f(x) - xu'(x) - xu(x) + \int_0^x (x^2 - t)u(t)dt, \\ u(0) = 0, u'(0) = 1. \end{cases}$$

$$u''(0) = -1,$$

$$\begin{cases} u'''(x) = (f(x))' - (xu'(x))' - (xu(x))' \\ + \left(\int_0^x (x^2 - t)u(t)dt \right)', \\ u(0) = 0, u'(0) = 1, u''(0) = 0. \end{cases}$$

Which is approximated by:

$$u(x_{i+1}) \approx u(x_i) + hu'(x_i) + \frac{h^2}{2}u''(x_i) + \frac{h^3}{3!}u'''(x_i)$$

Table 11
Modified Taylor series method of order three for example 6

Val of x	Exact sol	Approx sol	Error
0,00e+00	0,00e+00	0,00e+00	0,00e+00
2,50e-01	3,21e-01	3,20e-01	8,20e-05
5,00e-01	8,24e-01	8,24e-01	9,48e-05
7,50e-01	1,58e+00	1,58e+00	5,85e-05
1,00e+00	2,71e+00	2,71e+00	0,00e+00

Source: The authors.

Regularization method using modified Simpson method:

$$\begin{cases} u''(x) + xu'(x) + xu(x) - \int_0^x (x^2 - t)u(t)dt = f(x), \\ u(0) = 0, u'(0) = 1. \end{cases}$$

$$f(x) = x^2 \cos x + x \sin x - x^2,$$

After integrating twice, we obtain a Volterra integral equation; using the modified Simpson method, we get the approximate solution.

Table 12
Regularization method using modified Simpson method for example 6

Val of x	Exact sol	Approx sol	Error
0,00e+00	0,00e+00	0,00e+00	0,00e+00
2,50e-01	2,47e-01	3,21e-01	7,86e-08
5,00e-01	4,79e-01	8,24e-01	1,42e-07
7,50e-01	6,81e-01	1,58e+00	1,97e-07
1,00e+00	8,41e-01	2,71e+00	2,55e-07

Source: The authors.

6 Conclusion

The initial value problem of second order with Volterra integral can be transformed into a Taylor series with the first method and by the regularization method into a Volterra integral equation of the second kind. Some numerical examples illustrated the performance of these new methods and its convergence.

References

- [1] A. Ansari, N. Ahmad, "Numerical accuracy Fredholm integro-differential equations by using Adomian decomposition method," *Bull. Cal. Math. Soc.* vol. 115, issue 5, pp. 567-578, 2003. <https://doi.org/10.46939/J.Sci.Arts-23.3-a05>
- [2] D. Elmaci, B. Savaseneril, "Euler polynomials method for solving linear integro-differential equations," *NTMSCI* vol. 9, issue 3, pp. 21-34, 2001. <http://dx.doi.org/10.20852/ntmsci.2021.450>
- [3] S. El-gendi, "Chebyshev Solution of Differential, Integral and Integro-Differential Equations," *The Comp. journal*, vol. 12, issue 3, pp. 282-287, Aug. 1969. <https://doi.org/10.1093/comjnl/12.3.282>
- [4] M. Gachpazan, A. Kerayechian, "Finite Element Method for solving Linear Volterra Integro-Differential Equations of the second kind," *J of Inf and Comp Sci* vol. 9, issue 4, pp. 289-297, Jan. 2014.
- [5] S. Mathankumar, D. Poornima, "An Approximate Solution for Fuzzy Volterra Integro-differential Equations using Variational Iteration Technique," *Adv in Dyna Sys and Appl.* Vol. 16, issue 2, pp. 1455-1467, Dec. 2021.
- [6] M. Nadir, "A variational form with Bernoulli series for linear integral equations," *J. of Theo. and App. Comp. Sci.* vol. 8, issue 3, pp. 31-36, 2013. http://www.jtacs.org/archive/2014/3/3/JTACS_2014_03_03.pdf
- [7] M. Nadir, B. Lakehali, "A variational form with Legendre series for linear integral equations," *Malaya Journal of Matematik*, vol. 6, issue 1, pp. 49-52, jan. 2018. <https://doi.org/10.26637/MJM0601/0007>
- [8] M. Nadir, "Numerical solution of weakly singular integro-differential equations," *Malaya J. Mat.* Vol. 3, issue 2, pp. 187-190, Apr. 2015. <https://doi.org/10.26637/mjm302/008>
- [9] L. Rahmani, B. Rahimi, M. Mordad, "Numerical Solution of Volterra-Fredholm Integro-Differential Equation by Block Pulse Functions and Operational Matrices," *Gen. Math. Notes.* Vol. 4, issue 2, pp. 37-48, June. 2011.
- [10] R. Saadati, B. Raftari, "A Comparison Between the Variational Iteration Method and Trapezoidal Rule for Solving Linear Integro-Differential Equations," *World Applied Sciences Journal*, vol. 4, issue 3, pp. 321-325, 2008.
- [11] L. Zada, M. Al-Hamami, "A New Approach for Solving Fredholm Integro-Differential Equations," *Inf. Sci. Lett.* Vol. 10, issue 3, pp. 407-415, Sep. 2021. <http://dx.doi.org/10.18576/isl/100303>